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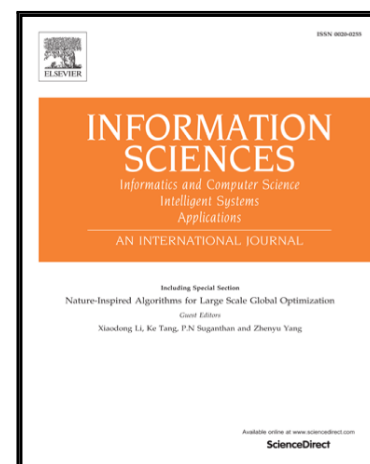
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Design of Polynomial Fuzzy Observer-Controller with Membership Functions using Unmeasurable Premise Variables for Nonlinear Systems

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Abstract

In this paper, the stability of polynomial fuzzy-model-based (PFMB) observer-control system is investigated via Lyapunov stability theory. The polynomial fuzzy observer with unmeasurable premise variables is designed to estimate the system states. Then the estimated system states are used for the state-feedback control of nonlinear systems. Although the consideration of the polynomial fuzzy model and unmeasurable premise variables enhances the applicability of the fuzzy-model-based (FMB) control strategy, it leads to non-convex stability conditions. Therefore, the refined completing square approach is proposed to derive convex stability conditions in the form of sum of squares (SOS) with less manually designed parameters. In addition, the membership functions of the polynomial observer-controller are optimized by the improved gradient descent method, which outperforms the widely applied parallel distributed compensation (PDC) approach according to a general performance index. Simulation examples are provided to verify the proposed design and optimization scheme.

Keywords: polynomial fuzzy observer-controller, optimized membership functions, unmeasurable premise variables, nonlinear system, sum of squares (SOS), gradient descent.

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1. Introduction

Stability analysis and control synthesis for nonlinear systems are difficult to be systematically conducted. Polynomial fuzzy model [51, 49] is one of the effective tools to model and analyze nonlinear systems, which is a generalization of Takagi-Sugeno (T-S) fuzzy model [45, 44] in terms of modeling capability. Both of them are employed in fuzzy-model-based (FMB) control strategies, which means that the stability analysis and control synthesis are carried out based on the fuzzy model instead of the nonlinear system [14]. Several techniques are widely employed under the FMB control scheme. First, the sector nonlinearity technique [50, 39] is exploited to represent the nonlinear system with the fuzzy model. Second, the Lyapunov stability theory [53] is applied to provide sufficient stability conditions. Third, linear matrix inequality (LMI) [46, 50] and sum of squares (SOS) approaches [36] are used to describe the stability conditions for the T-S fuzzy model and the polynomial fuzzy model, respectively, which can be solved by convex programming techniques. The SOS conditions can be converted into semidefinite programming problem by SOSTOOLS [35] and then solved by SeDuMi [41]. Furthermore, the parallel distributed compensation (PDC) [53] is implemented for the control synthesis. The feasibility of applying FMB control scheme, especially the polynomial fuzzy model and SOS approach, has been demonstrated by existing literature [47, 34, 9].

With respect to the development of FMB control strategy, the first task is to reduce the conservativeness of stability conditions. Three types of methods are investigated to deal with three sources of conservativeness, respectively. For the source of double fuzzy summation, Pólya's theory [37, 27] is exploited to offer progressively necessary and sufficient conditions which generalizes some earlier works [26, 10]. For the source of quadratic Lyapunov function, more general types of Lyapunov function candidates such as fuzzy Lyapunov function [29, 5, 24, 18], piecewise linear Lyapunov function [11, 12], switching Lyapunov function [32, 21] and polynomial Lyapunov function [4, 21] have been investigated

30 which include the quadratic one as a special case. For the source of membership-
function-independent stability conditions, the membership-function-dependent
approach is applied to make the stability conditions depend on membership
functions such as using approximated membership functions [30, 17], poly-
nomial constraints [38], symbolic variables [39, 22, 23] and other techniques
35 [3, 20, 16, 18, 7].

Another task of the development of FMB control strategy is to extend it to
solve control problems [40, 33, 42, 55, 43, 8, 15, 25, 54]. The T-S fuzzy observer
[46] has been extensively investigated to estimate the system states when the sys-
tem states are not measurable. Considering the case that the premise variables
40 of membership functions are measurable, one can easily apply the separation
principle [57] to design the fuzzy observer separately from the fuzzy controller.
However, in the case of unmeasurable premise variables, a two-step procedure
[31] was required due to the non-convex stability conditions. Since then, several
approaches have been proposed to achieve one-step design for unmeasurable
45 premise variables, for example, completing squares [13], matrix decoupling [52],
descriptor [6] and Finsler's lemma [1]. While the T-S fuzzy observer is widely
studied, the polynomial fuzzy observer receives relatively less attention. The
polynomial fuzzy observer was proposed in [48] which generalizes the T-S fuzzy
observer. The polynomial system matrices and polynomial input matrices are
50 allowed to exist in the polynomial fuzzy observer, and the observer gains can
also be polynomial. Nonetheless, the polynomial fuzzy observer-controller is
designed by two steps. The polynomial controller gains have to be obtained
first by assuming all system states are measurable. After that, the polynomial
observer gains can be subsequently determined. Moreover, only measurable
55 premise variables and constant output matrices are considered, which narrow
the applicability. To the best of our knowledge, the polynomial fuzzy observer-
controller with one-step design, unmeasurable premise variables and polynomial
output matrices has not been investigated.

Under the FMB control strategy, while the PDC approach is mainly em-
60 ployed to design the membership functions for the fuzzy observer-controller,

few works have been carried out to optimize the membership functions. Given a performance index (cost function) to evaluate the time response of the system, the membership functions from PDC approach may not be the optimal membership functions to offer the best time response. In [2], the optimal membership functions were designed under the frequency domain such that a desired closed-loop behavior is guaranteed throughout the entire operating domain. However, in some cases, only approximate optimal membership functions can be obtained. In [28], a systematic method for designing optimal membership functions was proposed in a general setting. The variational method is employed to acquire the gradient of the cost function with respect to design parameters in the membership functions, and the gradient descent approach is used to obtain the stationary point of the cost function. Nevertheless, the cost function does not take the control input into account, and the summation-one property of the membership functions is not considered resulting in imprecise calculation of the dynamics of the closed-loop system and the gradients. These limitations of the existing methods motivate us to investigate the optimization of membership functions for the fuzzy observer-controller.

In this paper, we aim to enhance the applicability of FMB control scheme by considering the polynomial fuzzy-model-based (PFMB) observer-controller. Compared with [48], we obtain the polynomial observer gains and controller gains in one step rather than two steps. The premise variables are unmeasurable which are more general than measurable premise variables, and the output matrices are allowed to be polynomial matrices instead of constant matrices. To achieve the one-step design, the completing square approach refining the one in [13] is employed to derive the convex stability conditions in terms of SOS. Compared with [13], the number of manually designed parameters is reduced from 4 to 3, and the polynomial fuzzy model considered in this paper is more general than the T-S fuzzy model. Moreover, we aim to improve the performance of the PFMB observer-control system by optimizing the membership functions of the polynomial fuzzy observer-controller. The optimal membership functions in this paper are understood in the following way: given a cost function, a set of lin-

ear (or polynomial) observer-controllers, and the form of membership function with some parameters to be optimized, the optimal membership functions are the ones that combine the linear observer-controllers to form a fuzzy observer-
 95 controller which provides the lowest cost subject to the system stability. The gradient descent approach improving the one in [28] is exploited to achieve the optimization, which provides better performance than PDC approach. Compared with [28], the observer-based system is considered in this paper and the cost function is generalized by taking into account the control input. More precise
 100 gradients are obtained by considering the summation-one property of the membership functions.

This paper is organized as follows. Some notations and the formulation of polynomial fuzzy model, polynomial fuzzy observer and polynomial fuzzy controller are presented in Section 2. Stability analysis of the PFMB observer-
 105 control system is conducted in Section 3. The optimization of membership functions of the polynomial observer-controller is carried out in Section 4. Simulation examples demonstrate the proposed design and optimization method in Section 5. Finally, a conclusion is drawn in Section 6.

2. Preliminary

2.1. Notation

The following notations are employed throughout this paper [36]. A monomial in $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ is a function of the form $x_1^{d_1}(t)x_2^{d_2}(t) \cdots x_n^{d_n}(t)$, where $d_i \geq 0, i = 1, 2, \dots, n$, are integers. The degree of a monomial is $d = \sum_{i=1}^n d_i$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is a finite linear combination of monomials with real coefficients. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is an SOS if it can be written
 115 as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^m \mathbf{q}_j(\mathbf{x}(t))^2$, where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial and m is a nonnegative integer. It can be concluded that if $\mathbf{p}(\mathbf{x}(t))$ is an SOS, then $\mathbf{p}(\mathbf{x}(t)) \geq 0$. The expressions of $\mathbf{M} > 0, \mathbf{M} \geq 0, \mathbf{M} < 0$ and $\mathbf{M} \leq 0$ denote the positive, semi-positive, negative and semi-negative definite matrices \mathbf{M} , respectively. The
 120 expression of $\mathbf{M}(\mathbf{x}(t))^T$ represents the transpose of $\mathbf{M}(\mathbf{x}(t))$. The symbol “*”

in a matrix represents the transposed element in the corresponding position. The symbol $\text{diag}\{\cdots\}$ stands for a block-diagonal matrix.

2.2. Polynomial Fuzzy Model

The polynomial fuzzy model for the nonlinear system is presented as follows [51]:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^p w_i(\mathbf{x}(t)) \left(\mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \right), \\ \mathbf{y}(t) &= \sum_{i=1}^p w_i(\mathbf{x}(t)) \mathbf{C}_i(\mathbf{x}(t))\mathbf{x}(t),\end{aligned}\quad (1)$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ is the state vector, and n is the dimension of the nonlinear system; p is the number of rules in the polynomial fuzzy model; $\mathbf{A}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times n}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times m}$ are the known polynomial system and input matrices, respectively; $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input vector; $\mathbf{y}(t) \in \mathbb{R}^l$ is the output vector; $\mathbf{C}_i(\mathbf{x}(t)) \in \mathbb{R}^{l \times n}$ is the polynomial output matrix; $w_i(\mathbf{x}(t))$ is the normalized grade of membership, $w_i(\mathbf{x}(t)) = \frac{\prod_{\eta=1}^{\Psi} \mu_{M_{\eta}^i}(f_{\eta}(\mathbf{x}(t)))}{\sum_{k=1}^p \prod_{\eta=1}^{\Psi} \mu_{M_{\eta}^k}(f_{\eta}(\mathbf{x}(t)))}$, $w_i(\mathbf{x}(t)) \geq 0, i = 1, 2, \dots, p$, and $\sum_{i=1}^p w_i(\mathbf{x}(t)) = 1$; $\mu_{M_{\eta}^i}(f_{\eta}(\mathbf{x}(t)))$, $\eta = 1, 2, \dots, \Psi$, are the grades of membership corresponding to the fuzzy term M_{η}^i ; $f_{\eta}(\mathbf{x}(t))$ is the premise variable corresponding to its fuzzy term M_{η}^i in rule i , $\eta = 1, 2, \dots, \Psi$, and Ψ is a positive integer.

2.3. Polynomial Fuzzy Observer

For brevity, time t is dropped from now. Define $\check{\mathbf{x}} \in \mathbb{R}^n$ as the estimated system state vector and $\check{\mathbf{y}} \in \mathbb{R}^l$ as the estimated system output vector. The following polynomial fuzzy observer is applied to estimate the states \mathbf{x} in (1):

$$\begin{aligned}\dot{\check{\mathbf{x}}} &= \sum_{i=1}^p m_i(\check{\mathbf{x}}) \left(\mathbf{A}_i(\check{\mathbf{x}})\check{\mathbf{x}} + \mathbf{B}_i(\check{\mathbf{x}})\mathbf{u} + \mathbf{L}_i(\check{\mathbf{x}})(\mathbf{y} - \check{\mathbf{y}}) \right), \\ \check{\mathbf{y}} &= \sum_{i=1}^p m_i(\check{\mathbf{x}}) \mathbf{C}_i(\check{\mathbf{x}})\check{\mathbf{x}},\end{aligned}\quad (2)$$

where $\mathbf{L}_i(\check{\mathbf{x}}) \in \mathbb{R}^{n \times l}$ is the polynomial observer gain; $m_i(\check{\mathbf{x}})$ is the membership function to be chosen and optimized, which satisfies $\sum_{i=1}^p m_i(\check{\mathbf{x}}) = 1$.

Remark 1. Since we consider unmeasurable premise variables $f_\eta(\mathbf{x})$ for the polynomial fuzzy model, the membership functions of the polynomial fuzzy observer $m_i(\check{\mathbf{x}})$ should be allowed to depend on estimated system states $\check{\mathbf{x}}$ rather than the original system states \mathbf{x} . Furthermore, the system output matrix $\mathbf{C}_i(\mathbf{x})$ is allowed to be a function of system states \mathbf{x} instead of constant matrix \mathbf{C}_i . The above settings include those in [48] as particular cases.

2.4. Polynomial Fuzzy Controller

With the obtained estimated system states $\check{\mathbf{x}}$ from (2), The polynomial fuzzy controller is described as follows:

$$\mathbf{u} = \sum_{i=1}^p m_i(\check{\mathbf{x}}) \mathbf{G}_i(\check{\mathbf{x}}) \check{\mathbf{x}}, \quad (3)$$

where $\mathbf{G}_i(\check{\mathbf{x}}) \in \mathbb{R}^{m \times n}$ is the polynomial controller gain.

Remark 2. The PDC approach with $m_i(\check{\mathbf{x}}) = w_i(\check{\mathbf{x}})$, $i = 1, 2, \dots, p$ is not necessarily applied in this paper. Instead, the membership function of the polynomial fuzzy observer-controller $m_i(\check{\mathbf{x}})$ is optimized such that the performance of the closed-loop system is better than PDC approach. Furthermore, the shapes of the membership function $m_i(\check{\mathbf{x}})$ can be chosen freely by users for different purposes. For example, the shapes can be chosen to be simpler than those of $w_i(\mathbf{x})$ to reduce the complexity of the observer-controller, or chosen to include the PDC approach as a special case for the comparison of performance during the optimization.

2.5. Useful Lemmas

The following lemmas will be employed in this paper.

Lemma 1. With \mathbf{X}, \mathbf{Y} of appropriate dimensions and $\gamma > 0$, the following inequality holds [56]:

$$\mathbf{X}^T \mathbf{Y} + \mathbf{Y}^T \mathbf{X} \leq \gamma \mathbf{X}^T \mathbf{X} + \frac{1}{\gamma} \mathbf{Y}^T \mathbf{Y}.$$

Lemma 2. *With \mathbf{P}, \mathbf{Q} of appropriate dimensions, $\mathbf{Q} > 0$ and a scalar γ , the following inequality holds [56]:*

$$-\mathbf{P}^T \mathbf{Q}^{-1} \mathbf{P} \leq \gamma^2 \mathbf{Q} - \gamma(\mathbf{P}^T + \mathbf{P}).$$

3. Stability Analysis

In this section, we conduct the stability analysis for PFMB observer-control systems. In the following, the dynamics of the closed-loop system is given first. Then, the stability conditions are derived based on the Lyapunov stability theory. The control synthesis is achieved by solving the stability conditions.

The estimation error is defined as $\mathbf{e} = \mathbf{x} - \check{\mathbf{x}}$, and then we have the closed-loop system consisting of the polynomial fuzzy model (1), the polynomial fuzzy controller (3) and the polynomial fuzzy observer (2) as follows:

$$\begin{aligned} \dot{\mathbf{x}} = & \sum_{i=1}^p \sum_{j=1}^p w_i(\mathbf{x}) m_j(\check{\mathbf{x}}) \left((\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\check{\mathbf{x}}) \mathbf{G}_j(\check{\mathbf{x}})) \mathbf{x} \right. \\ & \left. - \mathbf{B}_i(\mathbf{x}) \mathbf{G}_j(\check{\mathbf{x}}) \mathbf{e} \right), \end{aligned} \quad (4)$$

$$\begin{aligned} \dot{\check{\mathbf{x}}} = & \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p w_i(\mathbf{x}) m_j(\check{\mathbf{x}}) m_k(\check{\mathbf{x}}) \left((\mathbf{A}_j(\check{\mathbf{x}}) + \mathbf{B}_j(\check{\mathbf{x}}) \mathbf{G}_k(\check{\mathbf{x}}) \right. \\ & + \mathbf{L}_j(\check{\mathbf{x}}) (\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_k(\check{\mathbf{x}}))) \mathbf{x} + (-\mathbf{A}_j(\check{\mathbf{x}}) - \mathbf{B}_j(\check{\mathbf{x}}) \mathbf{G}_k(\check{\mathbf{x}}) \\ & \left. + \mathbf{L}_j(\check{\mathbf{x}}) \mathbf{C}_k(\check{\mathbf{x}})) \mathbf{e} \right), \end{aligned} \quad (5)$$

$$\begin{aligned} \dot{\mathbf{e}} = & \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p w_i(\mathbf{x}) m_j(\check{\mathbf{x}}) m_k(\check{\mathbf{x}}) \left((\mathbf{A}_i(\mathbf{x}) - \mathbf{A}_j(\check{\mathbf{x}}) \right. \\ & + (\mathbf{B}_i(\mathbf{x}) - \mathbf{B}_j(\check{\mathbf{x}})) \mathbf{G}_k(\check{\mathbf{x}}) - \mathbf{L}_j(\check{\mathbf{x}}) (\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_k(\check{\mathbf{x}}))) \mathbf{x} \\ & \left. + (\mathbf{A}_j(\check{\mathbf{x}}) - (\mathbf{B}_i(\mathbf{x}) - \mathbf{B}_j(\check{\mathbf{x}})) \mathbf{G}_k(\check{\mathbf{x}}) - \mathbf{L}_j(\check{\mathbf{x}}) \mathbf{C}_k(\mathbf{x})) \mathbf{e} \right). \end{aligned} \quad (6)$$

The control objective is to make the augmented PFMB observer-control system (formed by (4) and (6)) asymptotically stable, i.e., $\mathbf{x} \rightarrow 0$ and $\mathbf{e} \rightarrow 0$ as time $t \rightarrow \infty$, by determining the polynomial controller gain $\mathbf{G}_k(\check{\mathbf{x}})$ and polynomial observer gain $\mathbf{L}_j(\check{\mathbf{x}})$.

Theorem 1. *The augmented PFMB observer-control system (formed by (4) and (6)) is guaranteed to be asymptotically stable if there exist matrices $\mathbf{X} \in \mathbb{R}^{n \times n}$, $\mathbf{Y} \in \mathbb{R}^{n \times n}$, $\mathbf{N}_k(\check{\mathbf{x}}) \in \mathbb{R}^{m \times n}$, $\mathbf{M}_j(\check{\mathbf{x}}) \in \mathbb{R}^{n \times l}$, $k, j \in \{1, 2, \dots, p\}$ and pre-defined scalars $\gamma_1 > 0, \gamma_2 > 0, \gamma_3$ such that the following SOS-based conditions are satisfied:*

$$\nu_1^T(\mathbf{X} - \varepsilon_1 \mathbf{I})\nu_1 \text{ is SOS;} \quad (7)$$

$$\nu_2^T(\mathbf{Y} - \varepsilon_2 \mathbf{I})\nu_2 \text{ is SOS;} \quad (8)$$

$$-\nu_3^T(\Phi_{ijk}(\mathbf{x}, \check{\mathbf{x}}) + \Phi_{ikj}(\mathbf{x}, \check{\mathbf{x}}) + \varepsilon_3(\mathbf{x}, \check{\mathbf{x}})\mathbf{I})\nu_3 \text{ is SOS} \\ \forall i, j \leq k; \quad (9)$$

where

$$\Phi_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = \begin{bmatrix} \Theta_{ijk}(\mathbf{x}, \check{\mathbf{x}}) & \tilde{\Phi}^{(12)} & \tilde{\Phi}_j^{(13)}(\check{\mathbf{x}}) \\ * & -\frac{1}{\gamma_1} \mathbf{I} & \mathbf{0} \\ * & * & -\frac{1}{\gamma_2} \mathbf{I} \end{bmatrix}, \quad (10)$$

$$\Theta_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = \begin{bmatrix} \Gamma_{ijk}(\mathbf{x}, \check{\mathbf{x}}) & \tilde{\Theta}_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) & \tilde{\Theta}_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) & \Theta^{(14)} \\ * & -\gamma_1 \mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & * & -\gamma_2 \mathbf{I} & \mathbf{0} \\ * & * & * & \Theta_{jk}^{(44)}(\check{\mathbf{x}}) \end{bmatrix}, \quad (11)$$

$$\Gamma_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = \begin{bmatrix} \tilde{\Xi}_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) + \tilde{\Xi}_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}})^T & \tilde{\Xi}_{ik}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \\ * & -2\gamma_3 \mathbf{X} \end{bmatrix}, \quad (12)$$

$$\tilde{\Phi}^{(12)} = [\mathbf{0}_{n \times (3n+l)} \quad \mathbf{Y}]^T, \quad (13)$$

$$\tilde{\Phi}_j^{(13)}(\check{\mathbf{x}}) = [\mathbf{0}_{l \times (3n+l)} \quad \mathbf{M}_j(\check{\mathbf{x}})^T]^T, \quad (14)$$

$$\tilde{\Theta}_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) = [\tilde{\mathbf{H}}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) \quad \tilde{\mathbf{K}}_{ijk}(\mathbf{x}, \check{\mathbf{x}})]^T, \quad (15)$$

$$\tilde{\Theta}_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) = [(\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_k(\check{\mathbf{x}}))\mathbf{X} \quad \mathbf{0}_{l \times n}]^T, \quad (16)$$

$$\Theta^{(14)} = [\mathbf{0}_{n \times n} \quad \gamma_3 \mathbf{I}]^T, \quad (17)$$

$$\Theta_{jk}^{(44)}(\check{\mathbf{x}}) = \hat{\Xi}_{jk}^{(22)}(\check{\mathbf{x}}) + \hat{\Xi}_{jk}^{(22)}(\check{\mathbf{x}})^T, \quad (18)$$

$$\tilde{\Xi}_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) = \mathbf{A}_i(\mathbf{x})\mathbf{X} + \mathbf{B}_i(\mathbf{x})\mathbf{N}_k(\check{\mathbf{x}}), \quad (19)$$

$$\tilde{\Xi}_{ik}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) = -\mathbf{B}_i(\mathbf{x})\mathbf{N}_k(\check{\mathbf{x}}), \quad (20)$$

$$\hat{\Xi}_{jk}^{(22)}(\check{\mathbf{x}}) = \mathbf{Y}\mathbf{A}_j(\check{\mathbf{x}}) - \mathbf{M}_j(\check{\mathbf{x}})\mathbf{C}_k(\check{\mathbf{x}}), \quad (21)$$

$$\tilde{\mathbf{H}}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = (\mathbf{A}_i(\mathbf{x}) - \mathbf{A}_j(\check{\mathbf{x}}))\mathbf{X} + (\mathbf{B}_i(\mathbf{x}) - \mathbf{B}_j(\check{\mathbf{x}}))\mathbf{N}_k(\check{\mathbf{x}}), \quad (22)$$

$$\tilde{\mathbf{K}}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = -(\mathbf{B}_i(\mathbf{x}) - \mathbf{B}_j(\check{\mathbf{x}}))\mathbf{N}_k(\check{\mathbf{x}}); \quad (23)$$

ν_1, ν_2, ν_3 are arbitrary vectors independent of \mathbf{x} and $\check{\mathbf{x}}$ with appropriate dimensions; $\varepsilon_1 > 0, \varepsilon_2 > 0$ and $\varepsilon_3(\mathbf{x}, \check{\mathbf{x}}) > 0$ are predefined scalar polynomials; and the polynomial controller and observer gains are given by $\mathbf{G}_k(\check{\mathbf{x}}) = \mathbf{N}_k(\check{\mathbf{x}})\mathbf{X}^{-1}$ and $\mathbf{L}_j(\check{\mathbf{x}}) = \mathbf{Y}^{-1}\mathbf{M}_j(\check{\mathbf{x}})$, respectively. The number of decision variables is $n^2 + n + pn_t(mn + nl)$ where n_t is the the number of terms in each entry of the polynomial matrices $\mathbf{N}_k(\check{\mathbf{x}})$ and $\mathbf{M}_j(\check{\mathbf{x}})$. The number of SOS conditions is $\frac{1}{2}(p^3 + p^2) + 2$.

PROOF. Defining the augmented vector $\mathbf{z} = [\mathbf{x}^T \ \mathbf{e}^T]^T$ and the summation term $\sum_{i,j,k=1}^p W_{ijk} \equiv \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p w_i(\mathbf{x})m_j(\check{\mathbf{x}})m_k(\check{\mathbf{x}})$, the augmented PFMB observer-control system is written as

$$\dot{\mathbf{z}} = \sum_{i,j,k=1}^p W_{ijk} \Xi_{ijk}(\mathbf{x}, \check{\mathbf{x}})\mathbf{z}, \quad (24)$$

where

$$\Xi_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = \begin{bmatrix} \Xi_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) & \Xi_{ik}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \\ \Xi_{ijk}^{(21)}(\mathbf{x}, \check{\mathbf{x}}) + \mathbf{H}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) & \Xi_{jk}^{(22)}(\check{\mathbf{x}}) + \mathbf{K}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) \end{bmatrix}, \quad (25)$$

$$\Xi_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) = \mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_k(\check{\mathbf{x}}), \quad (26)$$

$$\Xi_{ijk}^{(21)}(\mathbf{x}, \check{\mathbf{x}}) = -\mathbf{L}_j(\check{\mathbf{x}})(\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_k(\check{\mathbf{x}})), \quad (27)$$

$$\Xi_{ik}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) = -\mathbf{B}_i(\mathbf{x})\mathbf{G}_k(\check{\mathbf{x}}), \quad (28)$$

$$\Xi_{jk}^{(22)}(\check{\mathbf{x}}) = \mathbf{A}_j(\check{\mathbf{x}}) - \mathbf{L}_j(\check{\mathbf{x}})\mathbf{C}_k(\check{\mathbf{x}}), \quad (29)$$

$$\mathbf{H}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = \mathbf{A}_i(\mathbf{x}) - \mathbf{A}_j(\check{\mathbf{x}}) + (\mathbf{B}_i(\mathbf{x}) - \mathbf{B}_j(\check{\mathbf{x}}))\mathbf{G}_k(\check{\mathbf{x}}), \quad (30)$$

$$\mathbf{K}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = -(\mathbf{B}_i(\mathbf{x}) - \mathbf{B}_j(\check{\mathbf{x}}))\mathbf{G}_k(\check{\mathbf{x}}). \quad (31)$$

The following Lyapunov function candidate is employed to investigate the stability of the augmented PFMB observer-control system (24):

$$V(\mathbf{z}) = \mathbf{z}^T \mathbf{P} \mathbf{z}, \quad (32)$$

where $\mathbf{P} = \begin{bmatrix} \mathbf{X}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix}$, $\mathbf{X} > 0$, $\mathbf{Y} > 0$, and thus $\mathbf{P} > 0$.

The time derivative of $V(\mathbf{z})$ is

$$\dot{V}(\mathbf{z}) = \sum_{i,j,k=1}^p W_{ijk} \mathbf{z}^T (\mathbf{P} \Xi_{ijk}(\mathbf{x}, \check{\mathbf{x}}) + \Xi_{ijk}(\mathbf{x}, \check{\mathbf{x}})^T \mathbf{P}) \mathbf{z}. \quad (33)$$

Therefore, $\dot{V}(\mathbf{z}) < 0$ holds if (the conservativeness is introduced)

$$\sum_{i,j,k=1}^p W_{ijk} (\mathbf{P} \Xi_{ijk}(\mathbf{x}, \check{\mathbf{x}}) + \Xi_{ijk}(\mathbf{x}, \check{\mathbf{x}})^T \mathbf{P}) < 0. \quad (34)$$

The augmented PFMB observer-control system (24) is guaranteed to be asymptotically stable if $V(\mathbf{z}) > 0$ by satisfying $\mathbf{P} > 0$ and $\dot{V}(\mathbf{z}) < 0$ by satisfying (34) excluding $\mathbf{x} = \mathbf{0}$. However, the condition (34) is not convex, which cannot be solved by convex programming technique. In what follows, we apply the refined completing square approach (Lemmas 1 and 2) and congruence transformation to derive (conservatively) convex SOS conditions such that the polynomial controller gain $\mathbf{G}_k(\check{\mathbf{x}})$ and the polynomial observer gain $\mathbf{L}_j(\check{\mathbf{x}})$ can be obtained in one step.

Denoting $\mathbf{M}_j(\check{\mathbf{x}}) = \mathbf{Y} \mathbf{L}_j(\check{\mathbf{x}})$, (34) becomes

$$\sum_{i,j,k=1}^p W_{ijk} (\hat{\Xi}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) + \hat{\Xi}_{ijk}(\mathbf{x}, \check{\mathbf{x}})^T) < 0, \quad (35)$$

where

$$\hat{\Xi}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = \begin{bmatrix} \mathbf{X}^{-1} \Xi_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) & \mathbf{X}^{-1} \Xi_{ik}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \\ \hat{\Xi}_{ijk}^{(21)}(\mathbf{x}, \check{\mathbf{x}}) + \mathbf{Y} \mathbf{H}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) & \hat{\Xi}_{jk}^{(22)}(\check{\mathbf{x}}) + \mathbf{Y} \mathbf{K}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) \end{bmatrix}, \quad (36)$$

$$\hat{\Xi}_{ijk}^{(21)}(\mathbf{x}, \check{\mathbf{x}}) = -\mathbf{M}_j(\check{\mathbf{x}})(\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_k(\check{\mathbf{x}})), \quad (37)$$

and $\hat{\Xi}_{jk}^{(22)}(\check{\mathbf{x}})$ is defined in (21).

Applying Lemma 1, we have

$$\begin{aligned}
 & \sum_{i,j,k=1}^p W_{ijk} (\hat{\Xi}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) + \hat{\Xi}_{ijk}(\mathbf{x}, \check{\mathbf{x}})^T) \\
 &= \sum_{i,j,k=1}^p W_{ijk} \left(\Upsilon_{ijk}(\mathbf{x}, \check{\mathbf{x}}) + \Theta_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \Phi^{(12)T} \right. \\
 & \quad + \Phi^{(12)} \Theta_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}})^T + \Theta_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) \Phi_j^{(13)}(\check{\mathbf{x}})^T \\
 & \quad \left. + \Phi_j^{(13)}(\check{\mathbf{x}}) \Theta_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}})^T \right) \\
 &\leq \sum_{i,j,k=1}^p W_{ijk} \Upsilon_{ijk}(\mathbf{x}, \check{\mathbf{x}}) + \gamma_1 \Phi^{(12)} \Phi^{(12)T} \\
 & \quad + \frac{1}{\gamma_1} \left(\sum_{i,j,k=1}^p W_{ijk} \Theta_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \Theta_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \right)^T \\
 & \quad + \gamma_2 \left(\sum_{i,j,k=1}^p W_{ijk} \Phi_j^{(13)}(\check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \Phi_j^{(13)}(\check{\mathbf{x}}) \right)^T \\
 & \quad + \frac{1}{\gamma_2} \left(\sum_{i,j,k=1}^p W_{ijk} \Theta_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \Theta_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) \right)^T \\
 &= \sum_{i,j,k=1}^p W_{ijk} \hat{\Upsilon}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) + \\
 & \quad + \frac{1}{\gamma_1} \left(\sum_{i,j,k=1}^p W_{ijk} \Theta_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \Theta_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \right)^T \\
 & \quad + \frac{1}{\gamma_2} \left(\sum_{i,j,k=1}^p W_{ijk} \Theta_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \Theta_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) \right)^T, \tag{38}
 \end{aligned}$$

where

$$\begin{aligned}
 \Upsilon_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = & \begin{bmatrix} \Upsilon_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) & \mathbf{X}^{-1} \Xi_{ik}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \\ * & \hat{\Xi}_{jk}^{(22)}(\check{\mathbf{x}}) + \hat{\Xi}_{jk}^{(22)}(\check{\mathbf{x}})^T \end{bmatrix}, \tag{39}
 \end{aligned}$$

$$\Upsilon_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) = \mathbf{X}^{-1} \Xi_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) + (\mathbf{X}^{-1} \Xi_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}}))^T, \tag{40}$$

$$\Phi^{(12)} = [\mathbf{0}_{n \times n} \quad \mathbf{Y}]^T, \tag{41}$$

$$\Phi_j^{(13)}(\check{\mathbf{x}}) = [\mathbf{0}_{l \times n} \quad \mathbf{M}_j(\check{\mathbf{x}})^T]^T, \tag{42}$$

$$\Theta_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) = [\mathbf{H}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) \quad \mathbf{K}_{ijk}(\mathbf{x}, \check{\mathbf{x}})]^T, \quad (43)$$

$$\Theta_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) = [\mathbf{C}_i(\mathbf{x}) - \mathbf{C}_k(\check{\mathbf{x}}) \quad \mathbf{0}_{l \times n}]^T, \quad (44)$$

$$\hat{\Upsilon}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = \begin{bmatrix} \Upsilon_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) & \mathbf{X}^{-1} \Xi_{ik}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \\ * & \hat{\Upsilon}_{jk}^{(22)}(\check{\mathbf{x}}) \end{bmatrix}, \quad (45)$$

$$\begin{aligned} \hat{\Upsilon}_{jk}^{(22)}(\check{\mathbf{x}}) &= \hat{\Xi}_{jk}^{(22)}(\check{\mathbf{x}}) + \hat{\Xi}_{jk}^{(22)}(\check{\mathbf{x}})^T + \gamma_1 \mathbf{Y} \mathbf{Y} \\ &+ \gamma_2 \left(\sum_{i,j,k=1}^p W_{ijk} \mathbf{M}_j(\check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \mathbf{M}_j(\check{\mathbf{x}}) \right)^T, \end{aligned} \quad (46)$$

and γ_1 and γ_2 are positive scalars.

There are two purposes of applying Lemma 1. One is separating matrix \mathbf{Y} from other unknown matrices. Another is leaving some convex (or convex after Schur Complement) terms into $\hat{\Upsilon}_{jk}^{(22)}(\check{\mathbf{x}})$ in (46). Subsequently, the purpose of applying Lemma 2 is exactly to preserve the convex terms in $\hat{\Upsilon}_{jk}^{(22)}(\check{\mathbf{x}})$ from being affected by the following congruence transformation. When separating matrix \mathbf{Y} , other unknown matrices can all be grouped into $\Theta_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}})$ in (43) such that only one design parameter is required, which is the reason that the number of design parameters is less than that in [13]. Note that the conservativeness is introduced by Lemmas 1 and 2.

Performing congruence transformation to both sides of (38) by pre-multiplying and post-multiplying $\text{diag}\{\mathbf{X}, \mathbf{X}\}$ and denoting $\mathbf{N}_k(\check{\mathbf{x}}) = \mathbf{G}_k(\check{\mathbf{x}})\mathbf{X}$, then $\dot{V}(\mathbf{z}) < 0$ holds if

$$\begin{aligned} & \sum_{i,j,k=1}^p W_{ijk} \tilde{\Upsilon}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) \\ & + \frac{1}{\gamma_1} \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\Theta}_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\Theta}_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \right)^T \\ & + \frac{1}{\gamma_2} \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\Theta}_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\Theta}_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) \right)^T \\ & < 0, \end{aligned} \quad (47)$$

where

$$\tilde{\mathbf{\Upsilon}}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) = \begin{bmatrix} \tilde{\mathbf{\Xi}}_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) + \tilde{\mathbf{\Xi}}_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}})^T & \tilde{\mathbf{\Xi}}_{ik}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \\ * & \mathbf{X} \hat{\mathbf{\Upsilon}}_{jk}^{(22)}(\check{\mathbf{x}}) \mathbf{X} \end{bmatrix}, \quad (48)$$

and $\tilde{\mathbf{\Theta}}_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}})$, $\tilde{\mathbf{\Theta}}_{ijk}^{(13)}(\mathbf{x}, \check{\mathbf{x}})$, $\tilde{\mathbf{\Xi}}_{ik}^{(11)}(\mathbf{x}, \check{\mathbf{x}})$ and $\tilde{\mathbf{\Xi}}_{ik}^{(12)}(\mathbf{x}, \check{\mathbf{x}})$, are defined in (15), (16), (19) and (20), respectively.

By grouping terms with same membership functions, $\dot{V}(\mathbf{z}) < 0$ holds if

$$\begin{aligned} & \sum_{i,j,k=1}^p W_{ijk} \left(\tilde{\mathbf{\Upsilon}}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) + \tilde{\mathbf{\Upsilon}}_{ikj}(\mathbf{x}, \check{\mathbf{x}}) \right) \\ & + \frac{2}{\gamma_1} \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\mathbf{\Theta}}_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\mathbf{\Theta}}_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \right)^T \\ & + \frac{2}{\gamma_2} \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\mathbf{\Theta}}_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\mathbf{\Theta}}_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) \right)^T \\ & < 0. \end{aligned} \quad (49)$$

Applying Lemma 2 to the term $\mathbf{X}(\hat{\mathbf{\Upsilon}}_{jk}^{(22)}(\check{\mathbf{x}}) + \hat{\mathbf{\Upsilon}}_{kj}^{(22)}(\check{\mathbf{x}}))\mathbf{X}$ (the conservativeness is introduced), we have

$$\begin{aligned} & \mathbf{X}(\hat{\mathbf{\Upsilon}}_{jk}^{(22)}(\check{\mathbf{x}}) + \hat{\mathbf{\Upsilon}}_{kj}^{(22)}(\check{\mathbf{x}}))\mathbf{X} \\ & = 2\mathbf{X} \frac{\hat{\mathbf{\Upsilon}}_{jk}^{(22)}(\check{\mathbf{x}}) + \hat{\mathbf{\Upsilon}}_{kj}^{(22)}(\check{\mathbf{x}})}{2} \mathbf{X} \\ & \leq 2 \left(-\gamma_3^2 \left(\frac{\hat{\mathbf{\Upsilon}}_{jk}^{(22)}(\check{\mathbf{x}}) + \hat{\mathbf{\Upsilon}}_{kj}^{(22)}(\check{\mathbf{x}})}{2} \right)^{-1} - 2\gamma_3 \mathbf{X} \right), \end{aligned} \quad (50)$$

where γ_3 is an arbitrary scalar.

Then $\dot{V}(\mathbf{z}) < 0$ holds if

$$\begin{aligned} & \sum_{i,j,k=1}^p W_{ijk} \left(\mathbf{\Gamma}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) + \mathbf{\Gamma}_{ikj}(\mathbf{x}, \check{\mathbf{x}}) \right) \\ & - 2\mathbf{\Theta}^{(14)} \left(\frac{\hat{\mathbf{\Upsilon}}_{jk}^{(22)}(\check{\mathbf{x}}) + \hat{\mathbf{\Upsilon}}_{kj}^{(22)}(\check{\mathbf{x}})}{2} \right)^{-1} \mathbf{\Theta}^{(14)T} \\ & + \frac{2}{\gamma_1} \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\mathbf{\Theta}}_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\mathbf{\Theta}}_{ijk}^{(12)}(\mathbf{x}, \check{\mathbf{x}}) \right)^T \end{aligned}$$

$$\begin{aligned}
 & + \frac{2}{\gamma_2} \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\Theta}_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) \right) \left(\sum_{i,j,k=1}^p W_{ijk} \tilde{\Theta}_{ik}^{(13)}(\mathbf{x}, \check{\mathbf{x}}) \right)^T \\
 & < 0,
 \end{aligned} \tag{51}$$

where $\mathbf{\Gamma}_{ijk}(\mathbf{x}, \check{\mathbf{x}})$ and $\mathbf{\Theta}^{(14)}$ are defined in (12) and (17).

By Schur Complement, we have

$$\sum_{i,j,k=1}^p W_{ijk} (\mathbf{\Phi}_{ijk}(\mathbf{x}, \check{\mathbf{x}}) + \mathbf{\Phi}_{ikj}(\mathbf{x}, \check{\mathbf{x}})) < 0, \tag{52}$$

where $\mathbf{\Phi}_{ijk}(\mathbf{x}, \check{\mathbf{x}})$ is defined in (10).

Therefore, $\dot{V}(\mathbf{z}) < 0$ if condition (49) holds which can be achieved by satisfying condition (9). Note that the conservativeness is introduced [51, 36] by using SOS conditions. The proof is completed.

4. Optimization of Membership Functions

After designing the polynomial observer-controller gains from Section 3, the subsequent objective is to optimize the membership functions of the polynomial fuzzy observer-controller $m_i(\check{\mathbf{x}})$ in (2) and (3).

It is assumed that $0 \leq m_i(\check{\mathbf{x}}, \alpha_i) \leq 1$ is designed as any differentiable functions with respect to both $\check{\mathbf{x}}$ and α_i , where $\alpha_i = [\alpha_{i1} \ \alpha_{i2} \ \cdots \ \alpha_{iq_i}]^T$, $i = 1, 2, \dots, p-1$ (p is the number of fuzzy rules), are parameters to be optimized (*e.g.*, Gaussian membership functions with mean and standard deviation to be determined). Then all parameters to be optimized are denoted as $\alpha = [\alpha_1^T \ \alpha_2^T \ \cdots \ \alpha_{p-1}^T]^T$. It is noted that the last membership function is defined as $m_p(\check{\mathbf{x}}, \alpha_1, \dots, \alpha_{p-1}) = 1 - \sum_{i=1}^{p-1} m_i(\check{\mathbf{x}}, \alpha_i)$ such that the condition $\sum_{i=1}^p m_i(\check{\mathbf{x}}, \alpha_i) = 1$ is satisfied. For brevity, we denote $\alpha_p = f(\alpha_1, \dots, \alpha_{p-1})$ and $m_p(\check{\mathbf{x}}, \alpha_1, \dots, \alpha_{p-1}) = m_p(\check{\mathbf{x}}, \alpha_p)$.

The cost function to be minimized in this paper is defined in the following general form:

$$J(\alpha) = \int_0^{T_t} \varphi(\mathbf{x}(t), \check{\mathbf{x}}(t), \alpha) dt + \psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha), \tag{53}$$

where T_t is the total time; φ and ψ are any differentiable functions with respect to \mathbf{x} , $\check{\mathbf{x}}$ and α .

215 **Remark 3.** In (53), the term $\int_0^{T_t} \varphi(\mathbf{x}(t), \check{\mathbf{x}}(t), \alpha) dt$ reflects the performance throughout time 0 to T_t and the term $\psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha)$ addresses the final state of the system at time T_t . Since we consider the equilibrium point to be $\mathbf{x} = \mathbf{0}$, these two terms are normally chosen to be non-negative such that the minimum is $J(\alpha) = 0$ when $\mathbf{x} = \mathbf{0}$. Both of these two terms are functions of
220 $\mathbf{x}, \check{\mathbf{x}}$ and α such that the estimated states $\check{\mathbf{x}}$ and the control input $\mathbf{u}(\check{\mathbf{x}}, \alpha)$ are allowed to exist in the cost function, which are more general than [28].

The constraint of the optimization is the dynamics of the closed-loop system (4) and (5) which is rearranged as follows:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\check{\mathbf{x}}} \end{bmatrix} = \sum_{i=1}^p \sum_{j=1}^p m_i(\check{\mathbf{x}}, \alpha_i) m_j(\check{\mathbf{x}}, \alpha_j) \mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}}),$$

$$\mathbf{x}(0) = \mathbf{x}_0, \quad \check{\mathbf{x}}(0) = \check{\mathbf{x}}_0, \quad (54)$$

where $\mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}}) = \sum_{k=1}^p w_k(\mathbf{x}) \begin{bmatrix} \mathbf{g}_{ijk}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) \\ \mathbf{g}_{ijk}^{(21)}(\mathbf{x}, \check{\mathbf{x}}) \end{bmatrix}$, $\mathbf{g}_{ijk}^{(11)}(\mathbf{x}, \check{\mathbf{x}}) = \mathbf{A}_k(\mathbf{x})\mathbf{x} + \mathbf{B}_k(\mathbf{x})\mathbf{G}_i(\check{\mathbf{x}})\check{\mathbf{x}}$,
 $\mathbf{g}_{ijk}^{(21)}(\mathbf{x}, \check{\mathbf{x}}) = (\mathbf{A}_j(\check{\mathbf{x}}) + \mathbf{B}_j(\check{\mathbf{x}})\mathbf{G}_i(\check{\mathbf{x}}))\check{\mathbf{x}} + \mathbf{L}_j(\check{\mathbf{x}})(\mathbf{C}_k(\mathbf{x})\mathbf{x} - \mathbf{C}_i(\check{\mathbf{x}})\check{\mathbf{x}})$; polynomial
observer-controller gains $\mathbf{G}_i(\check{\mathbf{x}})$ and $\mathbf{L}_j(\check{\mathbf{x}})$ are obtained from Section 3. It is
225 also assumed that the initial condition \mathbf{x}_0 is known such that the optimization can be carried out offline.

Remark 4. Under the condition $\sum_{i=1}^p m_i(\check{\mathbf{x}}, \alpha_i) = 1$, the calculated dynamics of the PFMB system (54) is equivalent to the dynamics of the original nonlinear system. In [28], however, the calculated dynamics is different from the dynamics
230 of the original nonlinear system without considering the summation-one condition. Since the gradients will be calculated based on the obtained dynamics, the gradients calculated in this paper will be more precise than those in [28].

The task is to optimize α according to the given performance index (53) under the constraint (54). In what follows, we propose sufficient conditions for

the stationary points of the cost function, and then apply the gradient descent method to find the parameters achieving the local minimum.

Applying the Lagrange multiplier $\lambda(t) \in \mathbb{R}^{1 \times 2n}$ to combine the constraint (54) (rearranged as a zero term) into the cost function (53):

$$\begin{aligned} & \tilde{J}(\alpha, \lambda) \\ &= \int_0^{T_t} \left(\varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha) + \lambda \left(\sum_{i=1}^p \sum_{j=1}^p m_i(\check{\mathbf{x}}, \alpha_i) m_j(\check{\mathbf{x}}, \alpha_j) \mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}}) \right. \right. \\ & \quad \left. \left. - [\dot{\mathbf{x}}^T \quad \dot{\check{\mathbf{x}}}^T]^T \right) \right) dt + \psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha). \end{aligned} \quad (55)$$

Note that the constraint (54) is placed in the integration from time 0 to T_t such that λ can be determined to eliminate some unknown variables in the following.

Theorem 2. A stationary point of the cost function (55) is obtained when the parameters $\alpha = [\alpha_1^T \quad \alpha_2^T \quad \cdots \quad \alpha_{p-1}^T]^T$ (where $\alpha_i = [\alpha_{i1} \quad \alpha_{i2} \quad \cdots \quad \alpha_{iq_i}]^T, i = 1, 2, \dots, p-1$) are chosen such that

$$\begin{aligned} & \frac{\partial \tilde{J}(\alpha, \lambda)}{\partial \alpha_{kl}} \\ &= \int_0^{T_t} \left(\lambda \sum_{i=1}^p m_i(\check{\mathbf{x}}, \alpha_i) \left(\frac{\partial m_k(\check{\mathbf{x}}, \alpha_k)}{\partial \alpha_{kl}} (\mathbf{g}_{ik}(\mathbf{x}, \check{\mathbf{x}}) + \mathbf{g}_{ki}(\mathbf{x}, \check{\mathbf{x}})) \right. \right. \\ & \quad \left. \left. + \frac{\partial m_p(\check{\mathbf{x}}, \alpha_p)}{\partial \alpha_{kl}} (\mathbf{g}_{ip}(\mathbf{x}, \check{\mathbf{x}}) + \mathbf{g}_{pi}(\mathbf{x}, \check{\mathbf{x}})) \right) \right. \\ & \quad \left. + \frac{\partial \varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha)}{\partial \alpha_{kl}} \right) dt + \frac{\partial \psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha)}{\partial \alpha_{kl}} \\ &= 0, \quad \forall k = 1, 2, \dots, p-1, l = 1, 2, \dots, q_k, \end{aligned} \quad (56)$$

where \mathbf{x} and $\check{\mathbf{x}}$ are given by the constraint (54) and the Lagrange multiplier $\lambda(t)$ is chosen such that

$$\begin{aligned} \dot{\lambda} &= - \left[\frac{\partial \varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha)}{\partial \mathbf{x}} \quad \frac{\partial \varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha)}{\partial \check{\mathbf{x}}} \right] \\ & \quad - \lambda \sum_{i=1}^p \sum_{j=1}^p \left(m_i(\check{\mathbf{x}}, \alpha_i) m_j(\check{\mathbf{x}}, \alpha_j) \left[\frac{\partial \mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}})}{\partial \mathbf{x}} \quad \frac{\partial \mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}})}{\partial \check{\mathbf{x}}} \right] \right. \\ & \quad \left. + \mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}}) [\mathbf{0}_{1 \times n} \quad \frac{\partial m_i(\check{\mathbf{x}}, \alpha_i)}{\partial \check{\mathbf{x}}} m_j(\check{\mathbf{x}}, \alpha_j) \right. \right. \\ & \quad \left. \left. + \frac{\partial m_j(\check{\mathbf{x}}, \alpha_j)}{\partial \check{\mathbf{x}}} m_i(\check{\mathbf{x}}, \alpha_i) \right] \right), \end{aligned}$$

$$\lambda(T_t) = \left[\frac{\psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha)}{\partial \mathbf{x}} \quad \frac{\psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha)}{\partial \check{\mathbf{x}}} \right]. \quad (57)$$

PROOF. The variational method [28] is employed to obtain $\frac{\partial \tilde{J}(\alpha, \lambda)}{\partial \alpha_{kl}}$ in (56), since it is difficult to calculate the partial derivative directly. Denoting the perturbed parameters as $\alpha_\epsilon = \alpha + \epsilon \vec{\theta}_{kl} = [\alpha_1^T, \dots, \alpha_{k\epsilon}^T, \dots, \alpha_{p-1}^T]^T$, where $\epsilon \ll 1$ and $\vec{\theta}_{kl} = [0, \dots, 0, \theta_{kl}, 0, \dots, 0]^T, k = 1, 2, \dots, p-1, l = 1, 2, \dots, q_k$, the resulting variation in the dynamics of the system becomes $\mathbf{x}_\epsilon = \mathbf{x} + \epsilon \eta_1(t)$ and $\check{\mathbf{x}}_\epsilon = \check{\mathbf{x}} + \epsilon \eta_2(t)$. Note that in parameters α_ϵ , only the l^{th} entry of $\alpha_{k\epsilon}$ is perturbed. Also, $\eta_1(0) = \eta_2(0) = \mathbf{0}$ since the initial conditions $\mathbf{x}_\epsilon(0) = \mathbf{x}(0) = \mathbf{x}_0, \check{\mathbf{x}}_\epsilon(0) = \check{\mathbf{x}}(0) = \check{\mathbf{x}}_0$ are unchanged. For brevity, we denote $\alpha_{p\epsilon} = f(\alpha_1, \dots, \alpha_{k\epsilon}, \dots, \alpha_{p-1})$. Therefore, the perturbed cost function is

$$\begin{aligned} & \tilde{J}_\epsilon(\alpha_\epsilon, \lambda) \\ &= \int_0^{T_t} \left(\varphi(\mathbf{x}_\epsilon, \check{\mathbf{x}}_\epsilon, \alpha_\epsilon) + \lambda(m_1(\check{\mathbf{x}}_\epsilon, \alpha_1)m_1(\check{\mathbf{x}}_\epsilon, \alpha_1)\mathbf{g}_{11}(\mathbf{x}_\epsilon, \check{\mathbf{x}}_\epsilon) \right. \\ & \quad + \dots + m_1(\check{\mathbf{x}}_\epsilon, \alpha_1)m_k(\check{\mathbf{x}}_\epsilon, \alpha_{k\epsilon})\mathbf{g}_{1k}(\mathbf{x}_\epsilon, \check{\mathbf{x}}_\epsilon) \\ & \quad + \dots + m_p(\check{\mathbf{x}}_\epsilon, \alpha_{p\epsilon})m_p(\check{\mathbf{x}}_\epsilon, \alpha_{p\epsilon})\mathbf{g}_{pp}(\mathbf{x}_\epsilon, \check{\mathbf{x}}_\epsilon) \\ & \quad \left. - [\dot{\mathbf{x}}_\epsilon^T \quad \dot{\check{\mathbf{x}}}_\epsilon^T]^T \right) dt + \psi(\mathbf{x}_\epsilon(T_t), \check{\mathbf{x}}_\epsilon(T_t), \alpha_\epsilon). \end{aligned} \quad (58)$$

Taking the directional derivative of $\tilde{J}(\alpha, \lambda)$ along the direction $\vec{\theta}_{kl}$, we have

$$\begin{aligned} & \nabla_{\vec{\theta}_{kl}} \tilde{J}(\alpha, \lambda) \\ &= \lim_{\epsilon \rightarrow 0} \frac{\tilde{J}_\epsilon(\alpha_\epsilon, \lambda) - \tilde{J}(\alpha, \lambda)}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\tilde{J}_\epsilon(\epsilon) - \tilde{J}_\epsilon(0)}{\epsilon - 0} \\ &= \left. \frac{d\tilde{J}_\epsilon(\epsilon)}{d\epsilon} \right|_{\epsilon=0} \\ &= \int_0^{T_t} \left(\frac{\varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha)}{\partial \mathbf{x}} \eta_1 + \frac{\varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha)}{\partial \check{\mathbf{x}}} \eta_2 + \frac{\varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha)}{\partial \alpha_{kl}} \theta_{kl} \right. \\ & \quad + \lambda \sum_{i=1}^p \sum_{j=1}^p (m_i(\check{\mathbf{x}}, \alpha_i) m_j(\check{\mathbf{x}}, \alpha_j) \left(\frac{\mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}})}{\partial \mathbf{x}} \eta_1 \right. \\ & \quad + \frac{\mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}})}{\partial \check{\mathbf{x}}} \eta_2) + \mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}}) \left(\frac{\partial m_i(\check{\mathbf{x}}, \alpha_i)}{\partial \check{\mathbf{x}}} m_j(\check{\mathbf{x}}, \alpha_j) \right. \\ & \quad \left. \left. + \frac{\partial m_j(\check{\mathbf{x}}, \alpha_j)}{\partial \check{\mathbf{x}}} m_i(\check{\mathbf{x}}, \alpha_i) \right) \eta_2 \right) \end{aligned}$$

$$\begin{aligned}
 & + \lambda \sum_{i=1}^p m_i(\check{\mathbf{x}}, \alpha_i) \left(\frac{\partial m_k(\check{\mathbf{x}}, \alpha_k)}{\partial \alpha_{kl}} (\mathbf{g}_{ik}(\mathbf{x}, \check{\mathbf{x}}) + \mathbf{g}_{ki}(\mathbf{x}, \check{\mathbf{x}})) \right. \\
 & + \frac{\partial m_p(\check{\mathbf{x}}, \alpha_p)}{\partial \alpha_{kl}} (\mathbf{g}_{ip}(\mathbf{x}, \check{\mathbf{x}}) + \mathbf{g}_{pi}(\mathbf{x}, \check{\mathbf{x}})) \left. \right) \theta_{kl} \\
 & - \lambda [\dot{\eta}_1^T \quad \dot{\eta}_2^T]^T dt + \frac{\psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha)}{\partial \mathbf{x}} \eta_1(T_t) \\
 & + \frac{\psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha)}{\partial \check{\mathbf{x}}} \eta_2(T_t) + \frac{\psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha)}{\partial \alpha_{kl}} \theta_{kl}. \tag{59}
 \end{aligned}$$

In (59), to deal with $\int_0^{T_t} -\lambda [\dot{\eta}_1^T \quad \dot{\eta}_2^T]^T dt$, we exploit integration by parts. Defining $\eta = [\eta_1^T \quad \eta_2^T]^T$ and recalling that $\eta_1(0) = \eta_2(0) = \mathbf{0}$, we have

$$\begin{aligned}
 \int_0^{T_t} -\lambda [\dot{\eta}_1^T \quad \dot{\eta}_2^T]^T dt & = -(\lambda \eta) \Big|_0^{T_t} + \int_0^{T_t} \dot{\lambda} \eta dt \\
 & = -\lambda(T_t) \eta(T_t) + \int_0^{T_t} \dot{\lambda} \eta dt. \tag{60}
 \end{aligned}$$

Substituting (60) into (59) and grouping terms, we have

$$\begin{aligned}
 & \nabla_{\vec{\theta}_{kl}} \tilde{J}(\alpha, \lambda) \\
 & = \int_0^{T_t} \left(\left[\frac{\varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha)}{\partial \mathbf{x}} \quad \frac{\varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha)}{\partial \check{\mathbf{x}}} \right] \right. \\
 & + \lambda \sum_{i=1}^p \sum_{j=1}^p (m_i(\check{\mathbf{x}}, \alpha_i) m_j(\check{\mathbf{x}}, \alpha_j) \left[\frac{\mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}})}{\partial \mathbf{x}} \quad \frac{\mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}})}{\partial \check{\mathbf{x}}} \right] \\
 & + \mathbf{g}_{ij}(\mathbf{x}, \check{\mathbf{x}}) [\mathbf{0}_{1 \times n} \quad \frac{\partial m_i(\check{\mathbf{x}}, \alpha_i)}{\partial \check{\mathbf{x}}} m_j(\check{\mathbf{x}}, \alpha_j) + \frac{\partial m_j(\check{\mathbf{x}}, \alpha_j)}{\partial \check{\mathbf{x}}} \\
 & \times m_i(\check{\mathbf{x}}, \alpha_i)] + \dot{\lambda} \left. \right) \eta dt + \theta_{kl} \left(\int_0^{T_t} \left(\lambda \sum_{i=1}^p m_i(\check{\mathbf{x}}, \alpha_i) \right. \right. \\
 & \times \left(\frac{\partial m_k(\check{\mathbf{x}}, \alpha_k)}{\partial \alpha_{kl}} (\mathbf{g}_{ik}(\mathbf{x}, \check{\mathbf{x}}) + \mathbf{g}_{ki}(\mathbf{x}, \check{\mathbf{x}})) + \frac{\partial m_p(\check{\mathbf{x}}, \alpha_p)}{\partial \alpha_{kl}} \right. \\
 & \times (\mathbf{g}_{ip}(\mathbf{x}, \check{\mathbf{x}}) + \mathbf{g}_{pi}(\mathbf{x}, \check{\mathbf{x}})) + \left. \frac{\varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha)}{\partial \alpha_{kl}} \right) dt \\
 & + \left. \frac{\psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha)}{\partial \alpha_{kl}} \right) \\
 & + \left(\left[\frac{\psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha)}{\partial \mathbf{x}} \quad \frac{\psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha)}{\partial \check{\mathbf{x}}} \right] \right. \\
 & \left. \left. - \lambda(T_t) \right) \eta(T_t). \tag{61}
 \end{aligned}$$

To find the relation between $\nabla_{\vec{\theta}_{kl}} \tilde{J}(\alpha, \lambda)$ in (61) and $\frac{\partial \tilde{J}(\alpha, \lambda)}{\partial \alpha_{kl}}$ in (56), we

have

$$\begin{aligned}\nabla_{\vec{\theta}_{kl}} \tilde{J}(\alpha, \lambda) &= \left. \frac{d\tilde{J}_\epsilon(\epsilon)}{d\epsilon} \right|_{\epsilon=0} \\ &= \left(\frac{\partial \tilde{J}_\epsilon(\alpha_\epsilon, \lambda)}{\partial \alpha_{kl}} \theta_{kl} \right) \Big|_{\epsilon=0} \\ &= \frac{\partial \tilde{J}(\alpha, \lambda)}{\partial \alpha_{kl}} \theta_{kl}.\end{aligned}\tag{62}$$

By choosing λ as in (57) and substituting (62) into (61), we can eliminate the unknown variables η and θ_{kl} , and obtain the expression for $\frac{\partial \tilde{J}(\alpha, \lambda)}{\partial \alpha_{kl}}$ as in (56). The proof is completed.

The following gradient descent algorithm [28] is employed to optimize the parameters α at each iteration i :

- 1) Compute \mathbf{x} and $\check{\mathbf{x}}$ forward from time 0 to T_t by (54).
- 2) Compute λ backward from time T_t to 0 by (57).
- 3) Compute the gradient $\nabla \tilde{J}(\alpha^{(i)}) = [\frac{\partial \tilde{J}(\alpha)}{\partial \alpha_{11}^{(i)}} \quad \frac{\partial \tilde{J}(\alpha)}{\partial \alpha_{12}^{(i)}} \quad \dots \quad \frac{\partial \tilde{J}(\alpha)}{\partial \alpha_{(p-1)q(p-1)}^{(i)}}]^T$ by (56).
- 4) Update the parameters $\alpha^{(i+1)} = \alpha^{(i)} - \beta^{(i)} \nabla \tilde{J}(\alpha^{(i)})$, where $\beta^{(i)}$ is the step size.

The algorithm terminates when the stopping criteria are met, for instance, the change of the gradient $|\nabla \tilde{J}(\alpha^{(i+1)}) - \nabla \tilde{J}(\alpha^{(i)})|$ is smaller than a limit or the maximum number of iterations is reached.

5. Simulation Examples

In this section, four examples are provided to show the procedure of applying the above design and optimization methods to control nonlinear systems. A numerical model is handled first, followed by three physical models.

5.1. Numerical Example

Consider the nonlinear system extended from [48]:

$$\begin{aligned}\dot{x}_1 &= \sin(x_1) + 5x_2 + (x_2^2 + 5)u, \\ \dot{x}_2 &= -x_1 - x_2^3, \\ y &= x_1 + 0.1x_1x_2^2.\end{aligned}$$

Defining the region of interest as $x_1 \in (-\infty, \infty)$, the nonlinear term $f_1(x_1) = \frac{\sin(x_1)}{x_1}$ is represented by sector nonlinearity technique [39] as follows: $f_1(x_1) = \mu_{M_1^1}(x_1)f_{1_{max}} + \mu_{M_1^2}(x_1)f_{1_{min}}$, where $\mu_{M_1^1}(x_1) = \frac{f_1(x_1) - f_{1_{min}}}{f_{1_{max}} - f_{1_{min}}}$, $\mu_{M_1^2}(x_1) = 1 - \mu_{M_1^1}(x_1)$, $f_{1_{min}} = -0.2172$, $f_{1_{max}} = 1.0000$. The system is exactly described by a 2-rule polynomial fuzzy model:

$$\begin{aligned}\dot{\mathbf{x}} &= \sum_{i=1}^2 w_i(x_1) (\mathbf{A}_i(x_2)\mathbf{x} + \mathbf{B}_i(x_2)u), \\ y &= \sum_{i=1}^2 w_i(x_1) \mathbf{C}_i(x_2)\mathbf{x},\end{aligned}$$

where $\mathbf{x} = [x_1 \ x_2]^T$; $\mathbf{A}_1(x_2) = \begin{bmatrix} f_{1_{max}} & 5 \\ -1 & -x_2^2 \end{bmatrix}$, $\mathbf{A}_2(x_2) = \begin{bmatrix} f_{1_{min}} & 5 \\ -1 & -x_2^2 \end{bmatrix}$, $\mathbf{B}_1(x_2) = \mathbf{B}_2(x_2) = [x_2^2 + 5 \ 0]^T$, and $\mathbf{C}_1(x_2) = \mathbf{C}_2(x_2) = [1 + 0.1x_2^2 \ 0]$;

the membership functions are $w_i(x_1) = \mu_{M_1^i}(x_1)$, $i = 1, 2$. It is assumed that both system states x_1 and x_2 are unmeasurable. Note that with the enhanced modeling capability of the polynomial fuzzy model, the polynomial term x_2^2 does not need to be modeled by the sector nonlinearity technique. Otherwise, 2 more rules are required and the only local stability in x_2 can be guaranteed.

Theorem 1 is employed to design the PFMB observer-controller to stabilize the system. We choose $\gamma_1 = 1 \times 10^{-3}$, $\gamma_2 = 1 \times 10^{-4}$, $\gamma_3 = 1$, $\mathbf{N}_k(\check{x}_2)$ of degree 0 and 2 in \check{x}_2 , $\mathbf{M}_j(\check{x}_2)$ of degree 0 and 2 in \check{x}_2 , and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1 \times 10^{-4}$. The polynomial controller gains are obtained as $\mathbf{G}_1(\check{x}_2) = [-1.7202 \times 10^{-1} \check{x}_2^2 - 3.5836 \times 10^{-1} \ - 6.0958 \times 10^{-2} \check{x}_2^2 - 3.0850 \times 10^{-1}]$ and $\mathbf{G}_2(\check{x}_2) = [-1.8171 \times 10^{-1} \check{x}_2^2 - 4.1202 \times 10^{-1} \ - 7.9720 \times 10^{-2} \check{x}_2^2 - 2.6510 \times 10^{-1}]$, and the polynomial observer gains are obtained as $\mathbf{L}_1(\check{x}_2) = [3.8483 \check{x}_2 + 6.7683 \ 1.2525 \check{x}_2 + 2.9268]^T$ and $\mathbf{L}_2(\check{x}_2) = [3.8713 \check{x}_2 + 5.6684 \ 1.2599 \check{x}_2 + 2.8682]^T$.

Remark 5. When users cannot manually determine the predefined parameters in Theorem 1 to find solutions, some algorithms such as genetic algorithm can be employed to search for feasible parameters. Moreover, less conservative form of the completing square approach can be applied, which however requires more predefined parameters.

To optimize the membership functions $m_i(\check{x}_1)$ of the polynomial fuzzy observer-controller, the Gaussian membership function is applied: $m_1(\check{x}_1, \alpha_1) = e^{-\frac{(\check{x}_1 - \alpha_{11})^2}{2\alpha_{12}^2}}$ and $m_2(\check{x}_1, \alpha_1) = 1 - m_1(\check{x}_1, \alpha_1)$, where $\alpha = [\alpha_1^T]^T = [\alpha_{11} \ \alpha_{12}]^T$ are the parameters to be optimized. We consider $\varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha) = \mathbf{x}^T \mathbf{Q} \mathbf{x} + u(\check{\mathbf{x}}, \alpha)^T R u(\check{\mathbf{x}}, \alpha)$, $\psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha) = \mathbf{x}(T_t)^T \mathbf{S} \mathbf{x}(T_t)$ in the cost function (53), where $\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 1$, $\mathbf{S} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$. The total time is $T_t = 10$ seconds, and the initial conditions are $\mathbf{x}_0 = [5 \ 0]^T$, $\check{\mathbf{x}}_0 = [0 \ 0]^T$. The stopping criterion is that the change of the gradient $|\nabla \tilde{J}(\alpha^{(i+1)}) - \nabla \tilde{J}(\alpha^{(i)})|$ is less than 0.01. Choosing the step size $\beta^{(i)} = 5$ (moderate step size should be chosen to avoid divergence and slow convergence speed) for all iterations i and initializing the parameters $\alpha^{(0)} = [0 \ 1]^T$, we obtain the optimized results $\alpha_{11} = 2.3137$, $\alpha_{12} = 1.1873$ and corresponding cost $J(\alpha) = 6.7519$. Comparing with the cost $J = 7.1428$ obtained by PDC approach ($m_i(\check{x}_1) = w_i(\check{x}_1)$, $i = 1, 2$), the optimized membership functions provide better performance.

To verify the optimized membership functions and cost, the gradient $\nabla J(\alpha)$ is shown in Fig. 1 generated by sampling parameters α . It can be seen that the lower costs occur when α_{11} is around 2.5 and α_{12} is around ± 1.5 , which coincides with the optimized parameters.

The original membership function $w_i(x_1)$ for the polynomial fuzzy model and the optimized membership function $m_i(\check{x}_1)$ for the polynomial fuzzy observer-controller are shown in Fig. 2(a) and Fig. 2(b), respectively. As shown in the figures, the optimized membership functions are different from the original membership function of the polynomial fuzzy model, which results in different

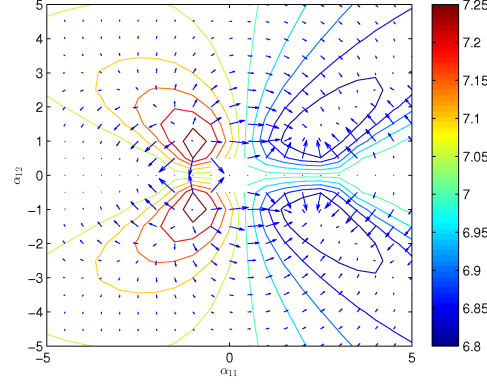
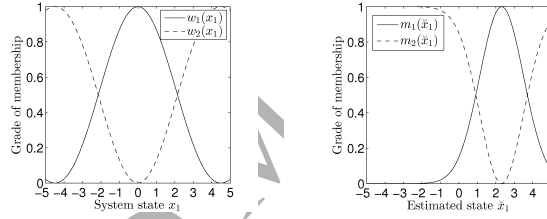


Figure 1: The descent of the gradient $\nabla J(\alpha)$, where the arrow indicates the direction of the gradient descent and the contour indicates the value of the cost $J(\alpha)$.



(a) $w_i(x_1)$ for the polynomial fuzzy model. (b) Optimized $m_i(\tilde{x}_1)$ for the polynomial fuzzy observer-controller.

Figure 2: Membership functions.

performance compared with the PDC approach. It is noted that the stability is still guaranteed since the previously employed positive and summation-one properties of membership functions remain unchanged.

Applying the designed polynomial observer-controller gains and the optimized membership functions to control the nonlinear system, the responses of system states, estimated states and their counterparts by PDC approach are shown in Fig. 3 and Fig. 4. The control input is shown in Fig. 5. The optimized membership functions perform better than the PDC approach with less overshoot and settling time.

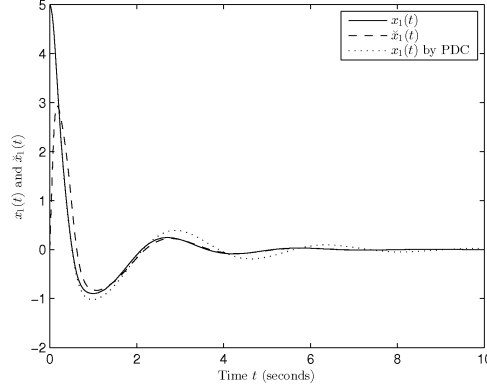


Figure 3: Time response of system state x_1 , its estimation \hat{x}_1 and its counterpart by PDC approach.

5.2. Nonlinear Mass-Spring-Damper System

Following the same procedure in Example 5.1, we try to stabilize a nonlinear mass-spring-damper system [19] with the following dynamics:

$$M\ddot{x} + g(x, \dot{x}) + f(x) = \phi(\dot{x})u,$$

where M is the mass; $g(x, \dot{x}) = D(c_1x + c_2\dot{x}^3 + c_3\dot{x})$, $f(x) = K(c_4x + c_5x^3 + c_6x)$ and $\phi(\dot{x}) = 1.4387 + c_7\dot{x}^2 + c_8 \cos(5\dot{x})$ are the damper nonlinearity, the spring nonlinearity and the input nonlinearity, respectively; $M = D = K = 1$, $c_1 = 0$, $c_2 = 1$, $c_3 = -0.3$, $c_4 = 0.01$, $c_5 = 0.1$, $c_6 = 0.3$, $c_7 = -0.03$, $c_8 = 0.2$; and u is the control input.

Time t is dropped from now for simplicity. Denoting x_1 and x_2 as x and \dot{x} , respectively, we obtain the following state space form:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{1}{M}(-g(x_1, x_2) - f(x_1) + \phi(x_2)u), \\ y &= x_1. \end{aligned}$$

The nonlinear term $f_1(x_2) = \cos(5x_2)$ is represented by sector nonlinearity technique [39] as follows: $f_1(x_2) = \mu_{M_1^1}(x_2)f_{1_{min}} + \mu_{M_1^2}(x_2)f_{1_{max}}$, where

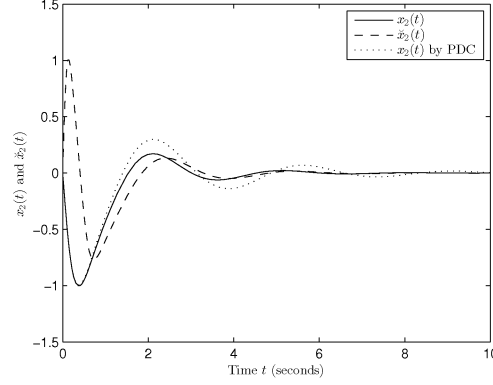


Figure 4: Time response of system state x_2 , its estimation \hat{x}_2 and its counterpart by PDC approach.

$\mu_{M_1^1}(x_2) = \frac{f_1(x_2) - f_{1_{max}}}{f_{1_{min}} - f_{1_{max}}}$, $\mu_{M_1^2}(x_2) = 1 - \mu_{M_1^1}(x_2)$, $f_{1_{min}} = -1$, $f_{1_{max}} = 1$. Therefore, the nonlinear mass-spring-damper system is precisely described by a 2-rule polynomial fuzzy model:

$$\begin{aligned}\dot{\mathbf{x}} &= \sum_{i=1}^2 w_i(x_2) (\mathbf{A}_i(\mathbf{x})\mathbf{x} + \mathbf{B}_i(x_2)u), \\ y &= \sum_{i=1}^2 w_i(x_2) \mathbf{C}_i \mathbf{x},\end{aligned}$$

where $\mathbf{x} = [x_1 \ x_2]^T$; $\mathbf{A}_1(\mathbf{x}) = \mathbf{A}_2(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ a_1(x_1) & a_2(x_2) \end{bmatrix}$, $a_1(x_1) = -\frac{1}{M}(Dc_1 + K(c_4 + c_6) + Kc_5x_1^2)$, $a_2(x_2) = -\frac{1}{M}(Dc_3 + Dc_2x_2^2)$; $\mathbf{B}_1(x_2) = [0 \ b_1(x_2)]^T$, $\mathbf{B}_2(x_2) = [0 \ b_2(x_2)]^T$, $b_1(x_2) = \frac{1}{M}(1.4387 + c_7x_2^2 + c_8f_{1_{min}})$, $b_2(x_2) = \frac{1}{M}(1.4387 + c_7x_2^2 + c_8f_{1_{max}})$; $\mathbf{C}_1 = \mathbf{C}_2 = [1 \ 0]$; the membership functions are $w_i(x_2) = \mu_{M_1^i}(x_2)$, $i = 1, 2$. Again, the polynomial fuzzy model demonstrates its superiority by keeping polynomial terms x_1^2 and x_2^2 . Otherwise, $2^3 = 8$ rules in total are required to precisely model the nonlinear mass-spring-damper system with only local stability in both x_1 and x_2 .

It is implied that the premise variable $f_1(x_2)$ depends on unmeasurable system state x_2 , and thus Theorem 1 is employed to design the PFMB observer-

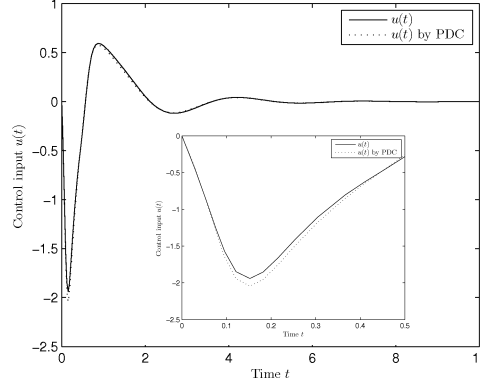


Figure 5: Time response of the control input u and its counterpart by PDC approach.

controller with unmeasurable premise variables. We choose $\gamma_1 = 1 \times 10^6$, $\gamma_2 = 1 \times 10^{-3}$, $\gamma_3 = 1 \times 10^{-2}$, $\mathbf{N}_k(\check{x}_1)$ of degree 0 and 2 in \check{x}_1 , $\mathbf{M}_j(\check{x}_1)$ of degree 0 and 2 in \check{x}_1 , $\varepsilon_1 = \varepsilon_2 = 1 \times 10^{-4}$, and $\varepsilon_3 = 1 \times 10^{-6}$. The polynomial controller gains are obtained as $\mathbf{G}_1(\check{x}_1) = [-4.3492 \times 10^{-1} \check{x}_1^2 - 8.3374 \times 10^{-2} \quad -2.7182 \check{x}_1^2 - 1.0842]$ and $\mathbf{G}_2(\check{x}_1) = [-4.2491 \times 10^{-1} \check{x}_1^2 - 2.8176 \times 10^{-1} \quad -2.7888 \check{x}_1^2 - 1.4408]$, and the polynomial observer gains are obtained as $\mathbf{L}_1(\check{x}_2) = [7.4229 \times 10^{-3} \check{x}_1^2 + 2.1987 \times 10^2 \quad 4.9731 \times 10^{-2} \check{x}_1^2 + 6.0260 \times 10^2]^T$ and $\mathbf{L}_2(\check{x}_2) = [7.4219 \times 10^{-3} \check{x}_1^2 + 2.1987 \times 10^2 \quad 4.9577 \times 10^{-2} \check{x}_1^2 + 6.0218 \times 10^2]^T$.

Remark 6. The existing polynomial observer [48] fails to deal with Examples 5.1 and 5.2 since it requires the premise variable to be measurable. To further compare with the two-step procedure in [48], we simplify the model in Example 5.2 by assuming the premise variable is measurable. However, by choosing the degree of polynomial matrix variables the same as those in this paper, no feasible solution is found. Consequently, the proposed one-step design is less conservative than the two-step procedure in [48].

To optimize the membership functions, in this example, we choose the sinusoidal membership function: $m_1(\check{x}_2, \alpha_1) = \frac{1}{2}(\sin(\alpha_{11}\check{x}_2 + \alpha_{12}) + 1)$ and $m_2(\check{x}_2, \alpha_1) = 1 - m_1(\check{x}_2, \alpha_1)$, where $\alpha = [\alpha_1^T]^T = [\alpha_{11} \quad \alpha_{12}]^T$ are the param-

ters to be optimized. The cost function, total time and stopping criteria are the same as in Example 5.1. The initial conditions are $\mathbf{x}_0 = [1 \ 0]^T$, $\check{\mathbf{x}}_0 = [0 \ 0]^T$. Choosing the step size $\beta^{(i)} = 2$ for all iterations i and initializing the parameters $\alpha^{(0)} = [0 \ 0]^T$, we obtain the optimized results $\alpha_{11} = 0.5347$, $\alpha_{12} = 0.5747$ and corresponding cost $J(\alpha) = 6.4968$, which is still better than the cost $J = 6.6349$ obtained by PDC approach ($m_i(\check{x}_2) = w_i(\check{x}_2)$, $i = 1, 2$).

To show the mechanism of the optimization, the descent of the gradient $\nabla J(\alpha)$ is shown in Fig. 6 and the original membership function $w_i(x_2)$ and the optimized membership function $m_i(\check{x}_2)$ are exhibited in Figs. 7(a) and 7(b), respectively. It can be summarized that the local minima appear periodically in terms of the phase α_{12} , which is consistent of the property of the sinusoidal function. The PDC approach is included in the optimization by considering $\alpha_{11} = 5$, $\alpha_{12} = -\frac{\pi}{2}$. As can be seen, the cost value of this point in Fig. 6 is larger than the one found by the optimization.

Remark 7. When the optimization is non-convex, the local minima may be found by the gradient descent approach instead of the global minima. Therefore, the resulting performance depends on the initial conditions of the optimization. However, a better performance than PDC approach can still be guaranteed by setting the initial condition of the optimization as the PDC approach, namely choosing the form of $m_i(\check{\mathbf{x}}, \alpha_i)$ and $\alpha^{(0)}$ such that $m_i(\check{\mathbf{x}}, \alpha_i) = w_i(\check{\mathbf{x}})$. In this way, the optimized performance is better than or at least equal to the PDC approach.

Applying the designed polynomial observer-controller gains and the optimized membership functions to control the nonlinear mass-spring-damping system, the responses of system states, estimated states and their counterparts by PDC approach are shown in Figs. 8 and 9. The response of the control input is shown in Fig. 10. Although the optimized membership functions lead to slightly more overshoot in x_1 , they save much more control energy in u . In other words, the optimization finds a better trade-off between the performance of the system states and the control energy, which results in a lower overall cost. In short, the

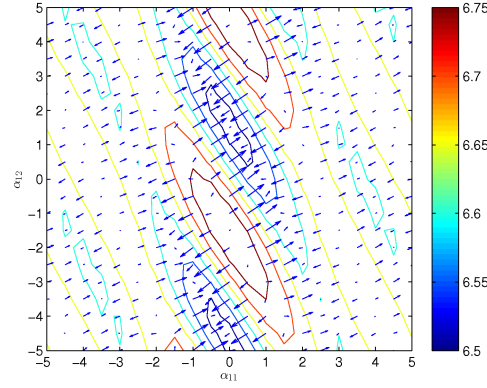
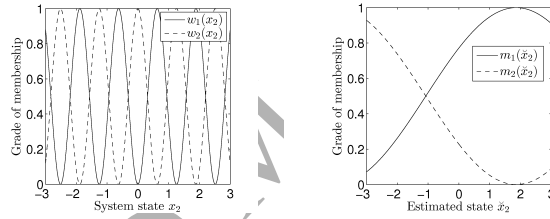


Figure 6: The descent of the gradient $\nabla J(\alpha)$, where the arrow indicates the direction of the gradient descent and the contour indicates the value of the cost $J(\alpha)$.



(a) $w_i(x_2)$ for the polynomial fuzzy model. (b) Optimized $m_i(\hat{x}_2)$ for the polynomial fuzzy observer-controller.

Figure 7: Membership functions.

proposed design and optimization of polynomial fuzzy observer-controller are feasible for controlling nonlinear systems.

5.3. Ball-and-Beam System

In this example, we further test the proposed approach on a system with higher dimension, namely the ball-and-beam system [19] with the following state-space form:

$$\dot{x}_1 = x_2,$$

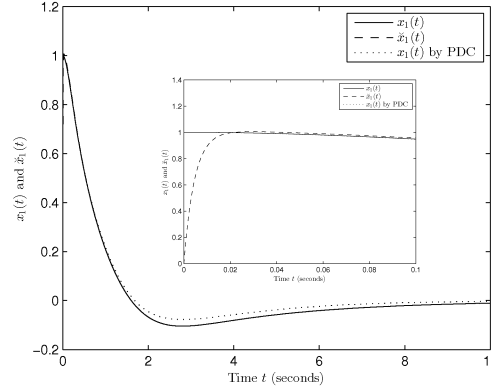


Figure 8: Time response of system state x_1 , its estimation \hat{x}_1 and its counterpart by PDC approach.

$$\dot{x}_2 = B(x_1 x_4^2 - g \sin(x_3)),$$

$$\dot{x}_3 = x_4,$$

$$\dot{x}_4 = u,$$

$$\mathbf{y} = [x_1 \ x_2 \ x_4]^T.$$

where x_1 and x_2 are the position and velocity of the ball, respectively; x_3 and x_4 are the angle and angular velocity of the beam, respectively; u is the control input; \mathbf{y} is the output vector; $B = 0.6$; $g = 10\text{m/s}^2$.

Defining the region of interest as $x_3 \in [-\frac{20\pi}{180}, \frac{20\pi}{180}]$, the nonlinear term $f_1(x_3) = \frac{\sin(x_3)}{x_3}$ is represented by sector nonlinearity technique [39] as follows: $f_1(x_3) = \mu_{M_1^1}(x_3)f_{1_{min}} + \mu_{M_1^2}(x_3)f_{1_{max}}$, where $\mu_{M_1^1}(x_3) = \frac{f_{1_{max}} - f_1(x_3)}{f_{1_{max}} - f_{1_{min}}}$, $\mu_{M_1^2}(x_3) = 1 - \mu_{M_1^1}(x_3)$, $f_{1_{min}} = 0.9798$, $f_{1_{max}} = 1.0000$. The system is exactly described by a 2-rule polynomial fuzzy model:

$$\begin{aligned} \dot{\mathbf{x}} &= \sum_{i=1}^2 w_i(x_3) (\mathbf{A}_i(x_4)\mathbf{x} + \mathbf{B}_i u), \\ y &= \sum_{i=1}^2 w_i(x_3) \mathbf{C}_i \mathbf{x}, \end{aligned}$$

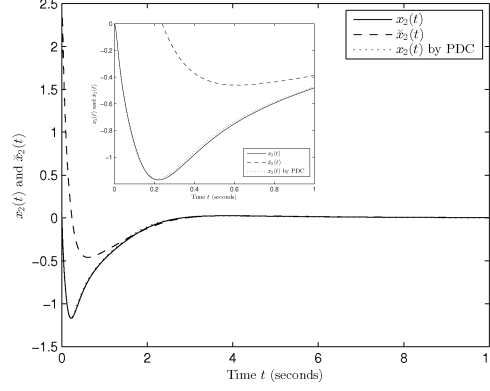


Figure 9: Time response of system state x_2 , its estimation \hat{x}_2 and its counterpart by PDC approach.

where

$$\mathbf{x} = [x_1 \quad x_2 \quad x_3 \quad x_4]^T,$$

$$\mathbf{A}_1(x_4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ Bx_4^2 & 0 & -Bgf_{1_{min}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{A}_2(x_4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ Bx_4^2 & 0 & -Bgf_{1_{max}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{B}_1 = \mathbf{B}_2 = [0 \quad 0 \quad 0 \quad 1]^T, \mathbf{C}_1 = \mathbf{C}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

the membership functions are $w_i(x_3) = \mu_{M_i^i}(x_3)$, $i = 1, 2$. Again, the polynomial fuzzy model demonstrates its superiority by keeping the polynomial term x_4^2 . Otherwise, $2^2 = 4$ rules are required by T-S fuzzy model in [19].

It is implied that the premise variable $f_1(x_3)$ depends on unmeasurable system state x_3 , and thus Theorem 1 is employed to design the PFMB observer-controller with unmeasurable premise variables. We choose $\gamma_1 = 1 \times 10^{-6}$, $\gamma_2 = 1 \times 10^{-2}$, $\gamma_3 = 2$, $\mathbf{N}_k(\check{x}_4)$ of degree 0 and 2 in \check{x}_4 , $\mathbf{M}_j(\check{x}_4)$ of degree 0 and 2 in \check{x}_4 , $\varepsilon_1 = \varepsilon_2 = 1 \times 10^{-4}$, and $\varepsilon_3 = 1 \times 10^{-6}$. The obtained polynomial

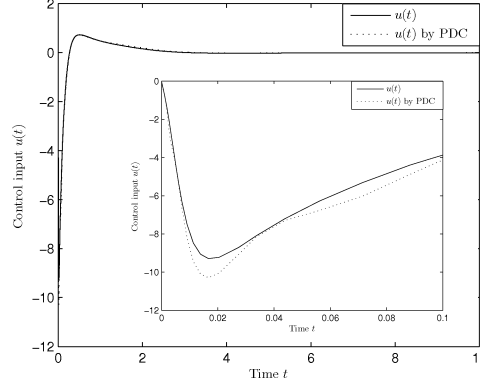


Figure 10: Time response of the control input u and its counterpart by PDC approach.

observer-controller gains are:

$$\begin{aligned} \mathbf{G}_1(\check{x}_4) &= [3.3079 \times 10^{-1} \check{x}_4^2 + 2.6902 \quad 7.6596 \times 10^{-2} \check{x}_4^2 + 2.5305 \\ &\quad - 2.1583 \times 10^{-1} \check{x}_4^2 - 1.0746 \times 10 \quad - 6.5337 \times 10^{-2} \check{x}_4^2 - 4.3596], \\ \mathbf{G}_2(\check{x}_4) &= [3.3078 \times 10^{-1} \check{x}_4^2 + 2.7261 \quad 7.6595 \times 10^{-2} \check{x}_4^2 + 2.5393 \\ &\quad - 2.1583 \times 10^{-1} \check{x}_4^2 - 1.0765 \times 10 \quad - 6.5337 \times 10^{-2} \check{x}_4^2 - 4.3655], \end{aligned}$$

$$\mathbf{L}_1(\check{x}_4) =$$

$$\begin{bmatrix} 2.16 \times 10^{-2} \check{x}_4^2 + 5.57 & 1.97 \times 10^{-1} \check{x}_4^2 + 5.18 \times 10 & -3.74 \times 10^{-4} \check{x}_4^2 - 2.11 \times 10^{-1} \\ 2.01 \times 10^{-1} \check{x}_4^2 + 5.07 \times 10 & 2.85 \check{x}_4^2 + 7.52 \times 10^2 & -4.99 \times 10^{-3} \check{x}_4^2 - 3.37 \\ -6.39 \times 10^{-3} \check{x}_4^2 - 1.55 & -1.14 \times 10^{-1} \check{x}_4^2 - 3.00 \times 10 & 8.18 \times 10^{-4} \check{x}_4^2 + 3.00 \times 10^{-1} \\ -2.95 \times 10^{-4} \check{x}_4^2 - 7.18 \times 10^{-2} & -5.16 \times 10^{-3} \check{x}_4^2 - 1.35 & 4.72 \times 10^{-5} \check{x}_4^2 + 1.62 \times 10^{-2} \end{bmatrix},$$

$$\mathbf{L}_2(\check{x}_4) =$$

$$\begin{bmatrix} 2.16 \times 10^{-2} \check{x}_4^2 + 5.57 & 1.97 \times 10^{-1} \check{x}_4^2 + 5.18 \times 10 & -3.74 \times 10^{-4} \check{x}_4^2 - 2.12 \times 10^{-1} \\ 2.01 \times 10^{-1} \check{x}_4^2 + 5.07 \times 10 & 2.85 \check{x}_4^2 + 7.52 \times 10^2 & -4.99 \times 10^{-3} \check{x}_4^2 - 3.39 \\ -6.39 \times 10^{-3} \check{x}_4^2 - 1.55 & -1.14 \times 10^{-1} \check{x}_4^2 - 3.00 \times 10 & 8.18 \times 10^{-4} \check{x}_4^2 + 3.01 \times 10^{-1} \\ -2.95 \times 10^{-4} \check{x}_4^2 - 7.18 \times 10^{-2} & -5.16 \times 10^{-3} \check{x}_4^2 - 1.35 & 4.72 \times 10^{-5} \check{x}_4^2 + 1.63 \times 10^{-2} \end{bmatrix}.$$

To optimize the membership functions $m_i(\check{x}_3)$ of the polynomial fuzzy observer-controller, the Gaussian membership function is applied: $m_1(\check{x}_3, \alpha_1) = e^{-\frac{(\check{x}_3 - \alpha_{11})^2}{2\alpha_{12}^2}}$

and $m_2(\check{x}_3, \alpha_1) = 1 - m_1(\check{x}_3, \alpha_1)$, where $\alpha = [\alpha_1^T]^T = [\alpha_{11} \quad \alpha_{12}]^T$ are the parameters to be optimized. The cost function, total time and stopping criteria are the

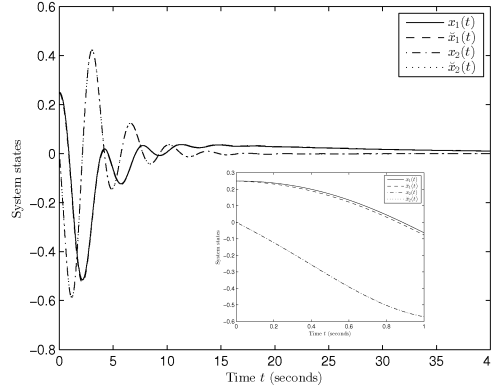


Figure 11: Time response of system state x_1 , its estimation \hat{x}_1 and its counterpart by PDC approach.

same as in Example 5.1. The initial conditions are $\mathbf{x}_0 = [0.25 \ 0 \ 0.1 \ 0]^T$, $\check{\mathbf{x}}_0 = [0.25 \ 0 \ 0 \ 0]^T$. Choosing the step size $\beta^{(i)} = 1$ for all iterations i and initializing the parameters $\alpha^{(0)} = [0.1 \ 0.1]^T$, we obtain the optimized results
 390 $\alpha_{11} = -0.0872, \alpha_{12} = 0.3147$ and the corresponding cost $J(\alpha) = 1.5068$, which is still better than the cost $J = 1.5266$ obtained by PDC approach ($m_i(\check{x}_3) = w_i(\check{x}_3), i = 1, 2$).

Applying the designed polynomial observer-controller gains and the optimized membership functions to control the ball-and-beam system, the responses
 395 of system states and estimated states are shown in Figs. 11 and 12. Again, the example demonstrates the applicability of the proposed design and optimization strategy.

Remark 8. The numerical complexity of applying Theorems 1 and 2 are shown in Tables 1 and 2, respectively. For Theorem 1, the computational time increases as the number of polynomial terms, the polynomial degrees, the dimension of the system and the number of fuzzy rules increase. As is known, the computational demand is relatively higher for the SOS technique compared with the LMI technique. For Theorem 2, the computational time also increases when the
 400

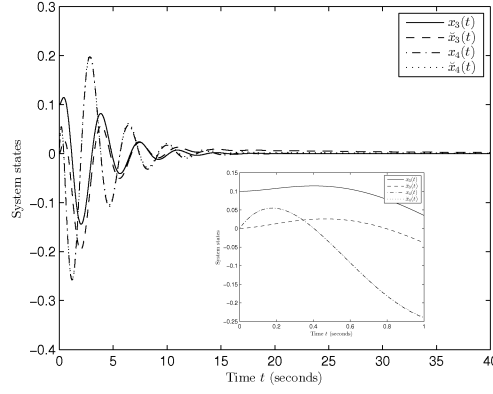


Figure 12: Time response of system state x_2 , its estimation \hat{x}_2 and its counterpart by PDC approach.

	Polynomial terms in model	Decision variables	SOS conditions	Computational time (minutes)
Example 5.1	1	22	8	1.6
Example 5.2	2	22	8	66.0
Example 5.3	1	84	8	21.4

Table 1: Numerical complexity of Theorem 1.

system is more complicated. This limitation makes the proposed optimization
 405 method only applicable offline instead of online.

5.4. Mobile Robot Navigation

In this example, we try to compare the proposed optimization scheme with
 the existing method in [28]. We consider the following unicycle model [28]:

$$\dot{x}_1 = v \cos(x_3),$$

$$\dot{x}_2 = v \sin(x_3),$$

$$\dot{x}_3 = u,$$

	Computational time (minutes/iteration)	Iterations
Example 5.1	4.5	6
Example 5.2	26.5	7
Example 5.3	92.1	4

Table 2: Numerical complexity of Theorem 2.

where (x_1, x_2) is the Cartesian coordinate of the center of the unicycle; $x_3 \in (-\pi, \pi]$ is its orientation with respect to the x_1 -axis; $v = 1$; u is the control input. Defining $\mathbf{x} = [x_1 \ x_2 \ x_3]^T$ and $\mathbf{z} = [x_1 \ x_2]^T$, the control objective is to navigate the mobile robot from initial position $\mathbf{x}_0 = [-1.5 \ 0 \ 0]^T$ to goal position $\mathbf{z}_g = [x_{1g} \ x_{2g}]^T = [3 \ 0]^T$ and avoid the obstacle $\mathbf{z}_a = [x_{1a} \ x_{2a}]^T = [0 \ 0]^T$.

Since the method in [28] cannot deal with fuzzy observer, we only employ fuzzy controller and assume all states are measurable. The fuzzy controller is given by:

$$u = \sum_{i=1}^2 m_i(x_1, \alpha_1) u_i,$$

where $m_1(x_1, \alpha_1) = 1 - e^{-\alpha_{11}(x_1 - x_{1a})^2}$ and $m_2(x_1, \alpha_1) = 1 - m_1(x_1, \alpha_1)$ are the membership functions with parameter $\alpha = [\alpha_1^T]^T = \alpha_{11}$ to be optimized; $u_1 = C_g(\phi_g(\mathbf{z}) - x_3)$ and $u_2 = C_a(\pi + \phi_a(\mathbf{z}) - x_3)$ are predefined control laws for behaviors “go-to-goal” and “avoid-obstacle”, respectively; $C_g = 10, C_a = 1$; $\phi_g(\mathbf{z}) = \arctan(x_{2g} - x_2, x_{1g} - x_1)$ and $\phi_a(\mathbf{z}) = \arctan(x_{2a} - x_2, x_{1a} - x_1)$ can be understood as angles from the goal position and the obstacle respectively to the robot when the robot is oriented to x_1 -axis.

We consider $\varphi(\mathbf{x}, \check{\mathbf{x}}, \alpha) = ae^{-b\|\mathbf{z} - \mathbf{z}_a\|^2} + c\|\mathbf{z} - \mathbf{z}_g\|^2$, $\psi(\mathbf{x}(T_t), \check{\mathbf{x}}(T_t), \alpha) = 0$ in the cost function (53), where $a = 2, b = 10, c = 0.01$. The first part $ae^{-b\|\mathbf{z} - \mathbf{z}_a\|^2}$ is used to drive the mobile robot away from the obstacle, and the second part $c\|\mathbf{z} - \mathbf{z}_g\|^2$ is used to drive the mobile robot to the goal position. The total time and stopping criteria are the same as in Example 5.1. Choosing the step size

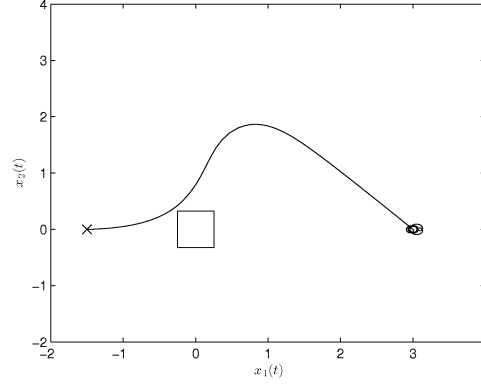


Figure 13: The trajectory of the mobile robot where “ \times ” indicates the initial position and “ \square ” indicates the obstacle position.

425 $\beta^{(i)} = 1$ for all iterations i and initializing the parameters $\alpha^{(0)} = 1$, we obtain the optimized results $\alpha_{11} = 0.7200$ and the trajectory of the mobile robot is shown in Fig. 13. Since the robot rotates and translates simultaneously, the robot does not move exactly towards the goal, which results in oscillation around the goal. The oscillation can be reduced by increasing the rotating coefficient C_g or decreasing the translating coefficient v .

Remark 9. The comparison with [28] is summarized in Table 3. The settings of [28] are the same as those in Example 5.4 except $m_2(x_1, \alpha_2) = e^{-\alpha_{21}(x_1 - x_{1a})^2}$, $u = \frac{\sum_{i=1}^2 m_i(x_1, \alpha_i) u_i}{\sum_{i=1}^2 m_i(x_1, \alpha_i)}$ and $\alpha_{11}, \alpha_{21} \in [0.1, 10]$. Using these settings, it can be seen that $m_2(x_1, \alpha_2)$ is independent of $m_1(x_1, \alpha_1)$ and thus $\sum_{i=1}^2 m_i(x_1, \alpha_i) \neq 1$ during the calculation of the gradient. Although the normalization is imposed on the final control signal $u = \frac{\sum_{i=1}^2 m_i(x_1, \alpha_i) u_i}{\sum_{i=1}^2 m_i(x_1, \alpha_i)}$, this is not considered in the algorithm and the calculated gradient is imprecise. Therefore, compared with existing approach, Theorem 2 provides more accurate gradient and less number of parameters to be optimized, which leads to lower cost and less computational time.

	Cost	Number of parameters	Computational time (minutes/iteration)	Iterations
Theorem 2	0.5590	1	1.5	10
[28]	0.6013	2	3.4	5

Table 3: Comparison of optimization algorithms.

6. Conclusion

In this paper, both the applicability and the performance of FMB control strategy have been improved. First, the polynomial fuzzy observer with unmeasurable premise variable has been designed based on the polynomial fuzzy model. Second, the membership functions of the polynomial observer-controller have been optimized to minimize a general performance index. The refined completing square approach and improved gradient descent method have been proposed to achieve the design and optimization, respectively. To draw a distinction from existing papers, more general settings (polynomial fuzzy model, unmeasurable premise variables and cost function), less design steps and parameters and more precise gradients have been attained in this paper. Simulation examples have been provided to demonstrate the enhanced applicability and performance. In the future, how to shorten the time of optimization to meet the requirement of online application can be further investigated. Also, the problems of applying polynomial Lyapunov function in the fuzzy observer-control system are left to be solved.

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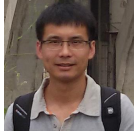
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